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Convergence of iteration method in the relativistic two-body problem, taking into account the retardation of interactions

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Abstract. A class of the Poincaré-invariant equations of motion (EOMs) of the two bodies is studied in the case of repulsion. The EOMs contain functional expressions depending upon the past history of the bodies. The convergence of an iteration method is proved which gives the 'instantaneous' equations describing all the weakly relativistic solutions of the initial EOMs. The results obtained rule out from a physical consideration the non-classical degrees of freedom introduced by the effects of heredity.

1. Introduction

In the investigations of dynamics of interacting bodies of a comparable mass one is often forced to eliminate field variables from the equations of motion (EOMs) in order to express these in terms of particle trajectories. Because of finite speed of propagation of interactions in relativistic dynamics the EOMs contain functional expressions depending upon the prehistory of the bodies, therefore the EOMs are delay-differential equations or, more generally, functional differential equations [1, 2], and not ordinary differential equations. The examples come from classical electrodynamics and from general relativity (see e.g. [3–8] and references therein). Besides difficulties in studying the EOMs, the dependence upon past history creates problems in formulating quantum mechanical and statistical models in terms of particle variables alone. In the weakly relativistic (WR) case (i.e. characteristic speed of particles $v_{ch} \ll 1$, where the speed of light = 1) one usually eliminates this dependence by means of truncated expansions in powers of v_{ch} or of an interaction constant, thus yielding a system of ordinary differential equations (SODE). The validity of such expansions is obscure, especially for the infinite time interval. The variety of solutions of a system of functional differential equations (SFDE) is usually much wider than that of SODE, and the infinity of solutions may be lost in the above approximation procedure [2]. As a result, one obtains only a finite number of degrees of freedom instead of the infinity of them typical in the presence of the effects of heredity [1, 2]. Therefore questions arise on the validity of the approximations involved and either the omitted degrees of freedom are of physical importance or they may be rejected. In this connection it would be instructive to analyse the questions for relatively simple exact EOMs allowing a rigorous treatment. This is carried out in this paper for a special class of Poincaré-invariant EOMs of two-point particles which take into account the retardation effects explicitly. At the same time the results on the convergence are valid for rather general EOMs. The main results are given by theorems 1 and 2.

The EOMs considered are described in section 2; they reflect the specificity of the relativistic dynamics and include in particular, the well-known exact equations which were studied earlier in the restricted (one-dimensional (1D) etc) cases [4-8]. We also discuss here the general questions concerning the statement of the problem. In section 3 we describe the iteration method and prove its convergence for a general SFDE. We state here the existence of an SODE with global solutions satisfying an initial SFDE. In section 4 these results are applied to the EOMs described in section 2. We show that the infinite number of degrees of freedom is ruled out if one is confined by the WR two-body trajectories considered up to the infinite past, these trajectories forming a phase flow of an 'instantaneous' SODE with the classical number of degrees of freedom.

2. Equations of motion

We consider the EOMs of two-point bodies described by the trajectories $x_p(t)$ corresponding to the one-particle action functionals (particularly for $p=1, 2$)

$$S_p = \int \mathrm{d}s_p(-m_p + gL_p) \tag{2.1}$$

where the masses $m_p > 0$ will be supposed to be of the same order, the constant g comprises the charges appropriate to the interactions involved, $\mathrm{d}s_p = (1-\dot{x}_p^2)^{1/2}\,\mathrm{d}t_p$

$$L_p = \frac{1}{2\pi}\int \mathrm{d}s_p\, \delta_\mathrm{D}[(t_p - t_q)^2 - (x_p - x_q)^2]\theta(t_p - t_q) f_{pq} \tag{2.1$'$}$$

where $p, q = 1, 2, p \neq q, x_p := x_p(t_p), x^\mu = (t, x)$ and $f_{pq} := f(P_{pq})$. $f(P)$ is the function supposed to be analytic in the neighbourhood of the point $P=1$, and $f(1)=1$,

$$P_{pq} = U_p^\mu \eta_{\mu\nu} U_q^\nu \qquad \{U_p^\mu\} := \left\{\frac{\mathrm{d}x_p^\mu}{\mathrm{d}s_p}\right\} := (U_p^0 U_p)$$

the Greek indexes run over 0-3, δ_D is the Dirac function, θ is the Heaviside step function, $\eta = \mathrm{diag}(1, -1, -1, -1)$. We consider the case of repulsion, i.e. $g > 0$.

The equations following from (2.1) as a result of variational principles $\delta S_p/\delta x_p = 0$ embrace the two-body EOMs of classical relativistic electrodynamics and those of the linear approximation of GR; the special cases have been studied in [4-8]. The explicit form of the EOMs

$$m_p\, \mathrm{d}^2 x_p(t)/\mathrm{d}t^2 = g\boldsymbol{H}_p(\boldsymbol{X}_t) \tag{2.2}$$

$(\boldsymbol{X}_t(u) := \boldsymbol{X}(t+u))$ is given by the relations

$$\boldsymbol{H}_p(x_t) := \boldsymbol{H}(x_p, x_q) = (1-\dot{x}_p^2)^{1/2}[\boldsymbol{A}_p - \dot{x}_p(\dot{x}_p \boldsymbol{A}_p)] \tag{2.2$'$}$$

$$\boldsymbol{A}_p := \partial L_p/\partial x_p - \frac{\mathrm{d}}{\mathrm{d}s}\{\partial[(1-\dot{x}_p^2)^{1/2} L_p]/\partial \dot{x}_p\}$$

$$= \frac{1}{2\pi}\int \mathrm{d}s_q\, \delta_\mathrm{D}[(t_p - t_q)^2 - (x_p - x_q)^2]\theta(t_p - t_q)$$

$$\times \left\{\frac{\mathrm{d}}{\mathrm{d}s_q}\left[\frac{x_p - x_q}{W_{qp}} f_{pq} + \frac{W_{pq}}{W_{qp}}\boldsymbol{B}_{pq}\right] - \frac{\mathrm{d}}{\mathrm{d}s_p}\boldsymbol{B}_{pq}\right\} \tag{2.2$''$}$$

where $\boldsymbol{B}_{pq} = \boldsymbol{U}_p[P_{pq} f'_{pq} - f_{pq}] - \boldsymbol{U}_q f'_{pq}$; $W_{pq} = (x_p^\mu - x_q^\mu) U_{p\mu}$, $t \equiv t_p$, $\mathrm{d}s \equiv \mathrm{d}s_p$.

The SFDE (2.2) is the system of differential equations of a neutral type (in the sense of [1]) with deviating arguments depending upon the unknown functions. The singularities of (2.2) correspond to collisions and light-like behaviour of the trajectories.

The fundamental effect of heredity introduced by the deviating arguments is that the solutions of the equations like (2.2) do not form the phase flow, that is, there may be the intersections of the trajectories in the phase space. As a rule these solutions are not specified uniquely by the initial conditions:

$$\dot{x}_p(t_0) = v_{p,0} \qquad x_p(t_0) = x_{p,0} \qquad p = 1, 2. \tag{2.3}$$

If the functions $x_p(t)$ are given on some segment of $t \leqslant t_0$, the unique continuous extension on $t > t_0$ exists for a rather general SFDE [1, 2]. Then, because the isolated *two-body system is subject for $t < t_0$ to the same* EOMs *as for $t > t_0$*, one comes to the 'backwards' problem [4] of solving (2.2) on $t \in (-\infty, t_0]$ under the conditions (2.3). There may still be an infinity of the backwards solutions satisfying (2.3); however, the main part of these had an unphysical behaviour in the past and may be ruled out. This is confirmed by the investigations of the 1D two-body problem of electrodynamics [4-8]. If the uniqueness of the backwards solutions from certain 'physical' classes does take place, one may hope that there exists an SODE

$$\mathrm{d}^2 x_p(t)/\mathrm{d}t^2 = G_p(\dot{x}_1(t), \dot{x}_2(t), x_1(t), x_2(t)) \tag{2.4}$$

$G_p : \mathbb{R}^{12} \to \mathbb{R}^3$, $p = 1, 2$, which generates these solutions. In this case SODE (2.4) may be referred to as an instantaneous form of the EOMs.

Thus the main questions are related to the existence of SODE (2.4), the convergence of the approximation method to construct G_p, and completeness of solutions of (2.4) to exhaust the physical solutions of SFDE (2.2).

We now give the precise formulation of the results concerning EOMs (2.2) to be proved in the following sections.

The functions $x_p \in C^2(\mathbb{R}, \mathbb{R}^3)$ will be said to be a solution of (2.2) if

$$|\dot{x}_p(t)| < 1 \qquad x_1(t) \neq x_2(t) \tag{2.5}$$

and they satisfy (2.2) identically on $\mathbb{R}$. Note that in the case of repulsion the solutions are continued on the whole $\mathbb{R}$; the restriction by the values $t < t_0$ would not change anything.

Define the domain $D(\varepsilon)$ from the space of variables

$$(v_1, v_2, x_1, x_2) \in \mathbb{R}^{12}$$

as follows:

$$D(\varepsilon) = \{(v_1, v_2, x_1, x_2) \colon v_p^2 \leqslant \varepsilon, p = 1, 2; k/|x_{12}| \leqslant \varepsilon\} \tag{2.6}$$

where $k = g/\min(m_1, m_2)$, $x_{12} := x_1 - x_2$, $\varepsilon > 0$ will be supposed to be sufficiently small. Let $W(\varepsilon)$ be a class of WR trajectories defined by the relations

$$x_p \in C^2(\mathbb{R}, \mathbb{R}^3) \qquad (\dot{x}_1(t), \dot{x}_2(t), x_1(t), x_2(t)) \in D(\varepsilon) \qquad \forall t \tag{2.7}$$

$$\sup\{|\ddot{x}_p(t)|, t \in \mathbb{R}\} < \infty \qquad p = 1, 2. \tag{2.8}$$

These properties are characteristic of any quasiclassical motion.

The following theorem proved in this paper answers the question of the existence of the instantaneous form of WR EOMs.

Theorem 1. In the EOMs (2.2) corresponding to (2.1) let $g>0$. There exist the values ε, $\varepsilon'(\varepsilon>\varepsilon'>0)$ and the Lipschitz-continuous functions $G_p:\mathbb{R}^{12}\to\mathbb{R}^3$, such that for any data $(v_{p,0}, x_{p,0})\in D(\varepsilon')$ there is a unique solution $\{x_p\}\in W(\varepsilon)$ of (2.2) with the conditions (2.3); this solution satisfies SODE (2.4).

3. The convergence of the iteration method

In this section we consider the convergence of iterations of the operator equations (3.20) corresponding to the the general n-dimensional SFDE (3.16) described below. The iteration procedure gives a sequence of SODE approximating the right-hand side of (3.16). The consideration relies upon work [9, 10] where it was shown that differential equations with sufficiently small retarded arguments admit the finitely parametric families of solutions as being the solutions of an SODE. We use the method of [10] to construct this SODE. The specificity of (2.2) is that these form neutral-type SFDE which contain the unbound retardations depending upon the unknown functions. This needs a modification of the results of [10], which has been carried out in our paper [11] used below to prove the convergence of the iterations of (3.20) yielding the exact 'instantaneous' SODE (3.23).

3.1. The properties of the functional classes involved

Consider the functions

$$\Delta_k:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}_+ \qquad k=1,\ldots,k_0$$

$$\Delta(y)=\max\{\Delta_k(y,y),\ k=1,\ldots,k_0\}$$

$$\mu_i:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}_+ \qquad 0<\mu_i(y,y)<\infty \qquad \forall y,y'\in\mathbb{R}^n$$

$$g_i,\ g_i^0 \qquad i=1,\ldots,n$$

$$Q:\mathbb{R}\to\mathbb{R}_+ \qquad g_i^1:\mathbb{R}\to\mathbb{R}_+ \qquad i=1,\ldots,n_0\leqslant n$$

satisfying the inequalities

$$\sup\left\{\sum_{i=1}^{n} g_i(y)\mu_i(y),\ y\in\mathbb{R}^n\right\}<\infty \tag{3.1}$$

$$g_i(y)\leqslant g_i^1(y) \qquad y\in\mathbb{R} \qquad i=1,\ldots,n_0 \tag{3.2}$$

$$g_i(z)\leqslant g_i^0(y) \qquad z\in d(y) \qquad i=1,\ldots,n \tag{3.3}$$

$$0<\varkappa_2\leqslant\mu_i(y,y')/\mu_i(z,z')\leqslant\varkappa_1<\infty \tag{3.4}$$

$$z\in d(y) \qquad z'\in d(y')$$

where $y=(y_1,\ldots,y_n)\in\mathbb{R}^n$, $\varkappa_1$ and $\varkappa_2$ are some constants, and $d(y)$ is the domain in $\mathbb{R}^n$:

$$d(y)=\{z=(z_1,\ldots,z_n)\in\mathbb{R}^n: |z_i-y_i|\leqslant\Delta(y)g_i^0(y),\ i=n_1,\ldots,n\} \qquad 1\leqslant n_1\leqslant n.$$

Define a set S of functions

$$X=(X_1,\ldots,X_n)\in C^1(J_{X(0)},\mathbb{R}^n) \qquad J_z:=\{s:|s|\leqslant\Delta(z)\}$$

satisfying the relations

$$|\dot{X}_i(s)| \leqslant g_i^1(X(s)) \qquad i=1,\ldots,n_0 \tag{3.5}$$

$$\sum_{i=1}^{n} \mu_i(X(0), X(0))|\dot{X}_i(s) - \dot{X}_i(s')| \leqslant |s-s'|Q(X(0)) \tag{3.6}$$

and the set $\boldsymbol{M}$ of function $F=(F_1,\ldots,F_n):\mathbb{R}^n \to \mathbb{R}^n$ such that

$$|F_i(y)| \leqslant g_i(y) \qquad i=1,\ldots,n \tag{3.7}$$

$$\sum_{i=1}^{n} \mu_i(y,y')|F_i(y)-F_i(y')| \leqslant L(y,y')n(y,y') \tag{3.8}$$

where

$$n(y,y') := \sum_{i=1}^{n} \mu_i(y,y')|y_i - y_i'|$$

and $\exists \sigma > 0$ such that

$$0 \leqslant L(z,z') \leqslant \sigma L(y,y') \qquad \text{for } z \in d(y) \qquad z' \in d(y') \tag{3.9}$$

$$Q(y) \geqslant \sigma_0 L(y,y) \sum_{i=1}^{n} \mu_i(y,y) g_i^0(y) \qquad \sigma_0 = \sigma\varkappa_1/\varkappa_2. \tag{3.10}$$

By virtue of (3.1) and (3.7) one may introduce the metric in $\boldsymbol{M}$:

$$m(F,F') = \sup\left\{\sum_{i=1}^{n} \mu_i(y,y)|F_i(y) - F_i'(y)|, y \in \mathbb{R}^n\right\}. \tag{3.11}$$

Lemma 1. On account of (3.2)–(3.4), (3.8) and (3.9) for any $F \in \boldsymbol{M}$ there exists a unique function $X_F(y,\cdot) \in \boldsymbol{S}$, satisfying the equation

$$X_F(y,t) = y + \int_0^t F(X_F(y,s))\,\mathrm{d}s \tag{3.12}$$

and for $|s| \leqslant \Delta(y)$ the following relations are valid:

$$X_F(y,s) \in d(y)$$

$$V_k(y,y',\delta X) \leqslant n(y,y') \exp[\sigma_0 L(y,y')\Delta_k(y,y')] \tag{3.13}$$

$$V_k(y,y',\sigma\dot{X}) \leqslant \sigma_0 n(y,y') L(y,y') \exp[\sigma_0(y,y')\Delta_k(y,y')] \tag{3.14}$$

where $\delta X := X_F(y,s) - X_F(y',s)$.

$$V_k(y,y',f) := \sum_{i=1}^{n} \mu_i(y,y') \sup\{|f_i(s)|, |s| \leqslant \Delta_k(y,y')\}. \tag{3.15}$$

Proof. For any $z, z' \in d(y)$ from (3.4), (3.8) and (3.9) it follows that

$$\sum_{i=1}^{n} |F(z) - F(z')|$$

$$\leqslant \varkappa_1 \mu_m \sum_{i=1}^{n} \mu_i(z,z')|F_i(z) - F_i(z')|$$

$$\leqslant \sigma_0 \mu_m L(y,y) \max\{\mu_k(y,y), k=1,\ldots,n\} \sum_{i=1}^{n} |Z_i - Z_i'|$$

where $\mu_m := [\min\{\mu_k(y,y), k=1,\ldots,n\}]^{-1}$.

Then by virtue of (3.8) and (3.9) F is Lipschitz-continuous in $d(y)$ and the existence of a unique solution $X_F=(X_1,\ldots,X_n)$ of (3.12), which does not leave $d(y)$ for $s\in J_y$, follows from (3.3). From (3.3) and (3.7) it follows

$$|X_i(y,s)-X_i(y,s')|\leqslant g_i^0(y)|s-s'|.$$

By virtue of (3.4), (3.8) and (3.9)

$$\sum_{i=1}^{n}\mu_i(y,y)|\dot{X}_i(y,s)-\dot{X}_i(y,s')|$$
$$\leqslant \sigma\varkappa_1 L(y,y)n(X(y,s),X(y,s'))$$
$$\leqslant \sigma_0 L(y,y)\sum_{i=1}^{n}\mu_i(y,y)|X_i(y,s)-X_i(y,s')|.$$

Making use of these estimates one can easily check conditions (3.5) and (3.6) for X_F. From (3.12) in view of (3.4), (3.8) and (3.9) it follows that

$$\sum_{i=1}^{n}\mu_i(y,y')|X_i(y,s)-X_i(y',s)|$$
$$\leqslant n(y,y')+\sigma_0 L(y,y')\sum_{i=1}^{n}\mu_i(y,y')\left|\int_0^s\left|X_i(y,u)-X_i(y',u)\right|\mathrm{d}u\right|.$$

This estimate leads to (3.13) by virtue of the Gronwall lemma. The inequality (3.14) follows from (3.4), (3.8), (3.9) and (3.13). □

3.2. *The functional differential system*

Consider the SFDE

$$\dot{X}(t)=\boldsymbol{H}(X_t) \tag{3.16}$$

where

$$X_t(s):=X(t+s)\qquad \boldsymbol{H}:=(\boldsymbol{H}_1,\ldots,\boldsymbol{H}_n):\boldsymbol{S}\to\mathbb{R}^n$$

and for $X, X'\in\boldsymbol{S}$

$$|\boldsymbol{H}_i(X)|\leqslant g_i(X(0)) \tag{3.17}$$

$$\sum_{i=1}^{n}\mu_i(X(0),X'(0))|\boldsymbol{H}_i(X)-\boldsymbol{H}_i(X')|$$
$$\leqslant L^0(X(0),X'(0))n(X(0),X'(0))$$
$$+\sum_{k=1}^{k_0}L_k^1(X(0),X'(0))V_k(X(0),X'(0),X-X')$$
$$+\sum_{k=1}^{k_0}L_k^2(X(0),X'(0))V_k(X(0),X'(0),\dot{X}-\dot{X}') \tag{3.18}$$

where

$$L^0, L_k^1, L_k^2:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}_+\qquad k=1,\ldots,k_0.$$

The function $X:\mathbb{R}\to\mathbb{R}^n$ will be said to be a solution of (3.16), if $X_t\in\boldsymbol{S}\ \forall t\in\mathbb{R}$ and X transforms (3.16) into identity on $\mathbb{R}$.

3.3. *Definition of the iteration procedure*

By virtue of lemma 1 the function $X_F \in \boldsymbol{S}\ \forall F \in \boldsymbol{M}$. Then on $\boldsymbol{M}$ there is defined the mapping $\boldsymbol{A}$:

$$\boldsymbol{A}(F)(y) = \boldsymbol{H}(X_F(y, \cdot)) \tag{3.19}$$

where $X_F(y, \cdot)$ is the solution of (3.12), $y \in \mathbb{R}^n$.

Conditions (3.3), (3.4) and (3.9) allows us to use the arguments analogous to that of [10] for estimating the iterations of the equation

$$F = \boldsymbol{A}(F) \tag{3.20}$$

to construct an SODE approximating (3.16). The sequence of functions F_n approximating the right-hand side of (3.16) is defined as $F_{n+1} = \boldsymbol{A}(F_n)$, $F_0 \equiv 0$.

3.4. *The contraction mapping properties of* $\boldsymbol{A}$

In the next step we prove $\{F_n\} \in \boldsymbol{M}$.

From condition (3.17) it is easy to see that the mapping $\boldsymbol{A}$ preserves the property (3.7). Making use of (3.13), (3.14) and (3.18) we find

$$\begin{aligned}\sum_{i=1}^{n} \mu_i(y, y')|\boldsymbol{H}_i(X_F(y, \cdot) - \boldsymbol{H}_i(X_F(y', \cdot))| \\ \leqslant n(y, y')\Bigg\{L^0(y, y') + \sum_{k=1}^{k_0} [\sigma_0 L(y, y')L_k^2(y, y') + L_k^1(y, y')] \\ \times \exp[\sigma_0 L(y, y')\Delta_k(y, y')]\Bigg\}\end{aligned}$$

whence there follows the next statement.

Lemma 2. Let the conditions of lemma 1 be fulfilled together with (3.17), (3.18) and

$$\sum_{k=1}^{k_0} [\sigma_0 L(y, y')L_k^2(y, y') + L_k^1(y, y')] \exp[\sigma_0 L(y, y')\Delta_k(y, y')] + L^0(y, y') \leqslant L(y, y')$$

$$y, y' \in \mathbb{R}^n \times \mathbb{R}^n \tag{3.21}$$

Then $\boldsymbol{A}(\boldsymbol{M}) \in \boldsymbol{M}$.

The following theorem shows the convergence of the sequence $\{F_n\} \in \boldsymbol{M}$.

Theorem 2. Let conditions (3.1)–(3.4), (3.9) and (3.10) be fulfilled and $\boldsymbol{H}$ satisfy conditions (3.17), (3.18) and (3.21) and

$$\sum_{k=1}^{k_0} [L_k^1(y, y)\Delta_k(y, y) + L_k^2(y, y)]\varkappa_1 \exp[\sigma_0 L(y, y)\Delta_k(y, y)] \leqslant q < 1 \tag{3.22}$$

q being a constant. Then

(i) there exists the unique $F \in \boldsymbol{M}$ satisfying (3.20);

(ii) any $X \in C^1(\mathbb{R}, \mathbb{R}^n)$ satisfying the SODE

$$\dot{X}(t) = F(X(t)) \qquad t \in \mathbb{R} \tag{3.23}$$

is the solution of (3.16).

Proof. Let m be given by (3.11). The pair $(\boldsymbol{M}, m)$ forms the complete metric space, which is mapped by the operator $\boldsymbol{A}$ into itself. Estimating the difference

$$\delta X(s) = X_F(y, s) - X_{F'}(y, s)$$

defined by (3.12) for $F, F' \in \boldsymbol{M}$ and using the estimates (3.4), (3.8) and (3.9) in analogy to lemma 1, on account of the Gronwall lemma one obtains

$$V_k(y, y, \delta X) \leqslant \{\exp[\sigma_0 L(y, y)\Delta_k(y, y)] - 1\}[\sigma_0 L(y, y)]^{-1} m(F, F')$$

$$V_k(y, y, \delta \dot{X}) \leqslant \varkappa_1 \exp[\sigma_0 L(y, y)\Delta_k(y, y)] m(F, F').$$

Substituting into (3.18) and using the inequality $e^z - 1 \leqslant z\, e^z$ for $z \geqslant 0$, under condition (3.22) one obtains

$$m(A(F), A(F')) \leqslant q m(F, F').$$

then statement (i) of the theorem follows from the contraction mapping principle. Statement (ii) follows from (3.20) and (3.12). □

4. Application to equations (2.2)

The consideration of (2.2) consists of the two parts. First, on account of the results of the previous section we prove the existence of the SODE (2.4) with WR solutions satisfying (2.2). Second, we prove that all WR solutions of (2.2) are uniquely defined by the data (2.3) from the region of asymptotically free motion. This yields that the WR solutions of (2.2) belong to the phase flow of the SODE (2.4) and, therefore, the uniqueness for the arbitrary data from $D(\varepsilon')$ with sufficiently small ε'.

4.1. The properties of the EOMs (2.2)

After calculations in accordance with (2.2′) and (2.2″) the EOMs (2.2) may be rewritten as

$$\mathrm{d}\boldsymbol{v}_p(t)/\mathrm{d}t = \boldsymbol{h}_p(X_t) \qquad \mathrm{d}\boldsymbol{x}_p(t)/\mathrm{d}t = \boldsymbol{v}_p(t) \tag{4.1}$$

where the functionals $\boldsymbol{h}_p(X)$ are defined on the 12-component functions $X(s) = \{\boldsymbol{v}_1(t), \boldsymbol{v}_2(t), \boldsymbol{x}_1(t), \boldsymbol{x}_2(t)\}$,

$$\begin{aligned} \boldsymbol{h}_p(X) = g m_p^{-1} \int & \mathrm{d}s\, \theta(s)\delta_{\mathrm{D}}[s^2 - |\boldsymbol{x}_p(t) - \boldsymbol{x}_q(t-s)|^2] \\ & \times \{\boldsymbol{K}_1[\boldsymbol{x}_p(t) - \boldsymbol{x}_q(t-s), \dot{\boldsymbol{x}}_p(t), \dot{\boldsymbol{x}}_q(t-s)] \\ & + \boldsymbol{K}_2[\boldsymbol{x}_p(t) - \boldsymbol{x}_q(t-s), \dot{\boldsymbol{x}}_p(t), \dot{\boldsymbol{x}}_q(t-s), \dot{\boldsymbol{v}}_p(t), \dot{\boldsymbol{v}}_q(t-s)]\} \\ & (p, q) = (1, 2), (2, 1). \end{aligned} \tag{4.2}$$

Here $\boldsymbol{K}_1[\boldsymbol{r}, \boldsymbol{u}_1, \boldsymbol{u}_2]$, $\boldsymbol{K}_2[\boldsymbol{r}, \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{w}_1, \boldsymbol{w}_2]$ are rational functions of $\boldsymbol{r}$, homogeneous upon $\boldsymbol{r}$ of the order of -1 and 0 respectively and analytic in $\boldsymbol{u}_p$ in the neighbourhood of zero, $\boldsymbol{K}_2$ being the linear homogeneous functions upon $\boldsymbol{w}_p$; both functions do not contain dimensional parameters; the relation

$$\boldsymbol{K}_1[\boldsymbol{r}, 0, 0] = \boldsymbol{r}/r^2 \qquad r = |\boldsymbol{r}| \tag{4.3}$$

gives the Newtonian limit. From the explicit form of these functions one obtains that

$$|\boldsymbol{K}_2[\boldsymbol{r}, \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{w}_1, \boldsymbol{w}_2]| < B(c_1)[|\boldsymbol{w}_1| + |\boldsymbol{w}_2|] \tag{4.4}$$

for $|u_p| \leqslant c_1 < 1$, where $B(\xi)$ is a continuous function upon ξ on $[0, 1)$.

On performing the integration in (4.2) there arise the retarded argiments $r_{pq}(t)$, $p \neq q$, defined by the equation

$$r_{pq} = |\boldsymbol{x}_p(t) - \boldsymbol{x}_q(t - r_{pq})|. \tag{4.5}$$

The equation (4.5) was first studied by R D Driver [12]; see also [4–8] and references therein. The necessary information on the properties of r_{pq} needed in the subsequent treatment is given by the following lemma.

Lemma 3. Let the functions $\boldsymbol{x}_i(t) \in c^2$ satisfy the conditions

$$|\dot{\boldsymbol{x}}_p(t)| < c_1 < 1 \qquad \boldsymbol{x}_1(t) \neq \boldsymbol{x}_2(t).$$

Then there is a unique solution $r_{pq}(t) \in C^2$ of (4.5), which has the following properties:

(i) $\quad (1 - c_1)/(1 + c_1) < |\mathrm{d}r_{pq}(t)/\mathrm{d}t| < (1 + c_1)/(1 - c_1)$

(ii) $\quad (1 - c_1) < |\boldsymbol{x}_p(t) - \boldsymbol{x}_q(t)|/r_{pq}(t) < (1 + c_1)$.

Under the variation of

$$\boldsymbol{x}_p(t) \to \boldsymbol{x}'_p(t) = \boldsymbol{x}_p(t) + \delta\boldsymbol{x}_p(t) \qquad i = 1, 2$$

the following estimate takes place:

(iii) $\quad |r_{pq}(t) - r'_{pq}(t)| < (1 - c_1)^{-1}\{|\delta\boldsymbol{x}_p(t)| + |\delta\boldsymbol{x}_q(t - r^m_{pq}(t))|\}$

where $r^m_{pq} = \min\{r_{pq}, r'_{pq}\}$.

The proof of statements (i) and (ii) is along the same lines as in [4–8, 12]. Let e.g.$r_{pq}(t) \leqslant r'_{pq}(t)$. Under the variation of $\boldsymbol{x}_p(t)$, (4.5) for r_{pq} and r'_{pq} yields

$$|r_{pq}(t) - r'_{pq}(t)| \leqslant |\delta\boldsymbol{x}_p(t)| + |\delta\boldsymbol{x}_q(t - r^m_{pq}(t))| + |\boldsymbol{x}'_q(t - r_{pq}) - \boldsymbol{x}'_q(t - r'_{pq})|$$
$$\leqslant |\delta\boldsymbol{x}_p(t)| + |\delta\boldsymbol{x}_q(t - r^m_{pq}(t))| + c_1|r_{pq}(t) - r'_{pq}(t)|$$

whence (iii) follows on account of $0 < c_1 < 1$.

4.2. *Applications of theorem 2 to the auxiliary* EOMs

Since (2.2) do not permit a direct application of theorem 2 beyond the functional domain $W(\varepsilon)$, we shall pass from these to the auxiliary truncated equations. The right-hand side of these is equivalent to the RHS of (2.2) on the WR trajectories, and it goes smoothly to zero beyong the WR region. Then using theorem 2 in the case of a sufficiently small ε we prove the existence of the SODE (2.4), its solutions satisfying the truncated equations.

Introduce the truncated equations

$$\mathrm{d}\boldsymbol{v}_p(t)/\mathrm{d}t = \boldsymbol{h}^*_p(X_t) \qquad \mathrm{d}\boldsymbol{x}_p(t)/\mathrm{d}t = \boldsymbol{p}(\boldsymbol{v}_p(t)) \tag{4.6}$$

where

$$\boldsymbol{h}^*_p(X_t) := T\{R_0/[2x_{12}(t)]\}\boldsymbol{h}_p(X_t) \qquad \boldsymbol{p}(\boldsymbol{v}) := \boldsymbol{v}T\{|\boldsymbol{v}|/(2c_1)\}$$

$$T\{\xi\} := \{1 \text{ for } \xi \in [0, 0.5]; 2(1 - \xi) \text{ for } [0.5, 1]; 0 \text{ for } \xi > 1\} \qquad R_0 = 0.$$

Evidently the solutions of (4.6) satisfying the relations

$$|\dot{x}_p(t)| < c_1 < 1 \qquad x_{12}(t) \geq R_0 \tag{4.7}$$

$x_{12} := |x_1 - x_2|$ for all possible t, will be also the solutions of (4.1).

Further we put $c_1 < \frac{1}{3}$.

To apply theorem 2 to EOMs (4.6) we admit the subsequent numeration for the 12-component vector $x = (v_1, v_2, x_1, x_2)$. In the conditions for S we put

$$\begin{aligned} g_i^1(x) &= kq_0(x_{12} + R_0)^{-2} \qquad i = 1, \ldots, 6 \\ g_i^1(x) &= 2c_1 \qquad i = 7, \ldots, 12 \end{aligned} \tag{4.8}$$

$$\begin{aligned} \mu_i(x, x') &= 1 \qquad i = 1, \ldots, 6 \\ \mu_i(x, x') &= [\bar{\Delta}(x, x')]^{-1} \qquad i = 7, \ldots, 12 \end{aligned} \tag{4.9}$$

where

$$\bar{\Delta}(x, x') = \Delta(x, x') + (1 - c_1)^{-1} R_0$$

$$\Delta(x, x') = (1 - c_1)^{-1} \min(x_{12}, x'_{12})$$

and

$$Q(x) = [\bar{\Delta}(x, x')]^{-2} \tag{4.10}$$

the constants Q_0, q_0 will be defined below. In the relations of the previous section we put $n = n_0 = 12$, $k_0 = 1$, $n_1 = 7$;

$$J_x = \{s : |s| \leq (1 - c_1)^{-1} x_{12}\}$$

$$\mathrm{d}(x) = \{x' = (v'_1, v'_2, x'_1, x'_2) \in \mathbb{R}^{12} : |x_i - x'_i| \leq b_1 x_{12}\}$$

$$b_1 = c_1/(1 - c_1).$$

According to (4.9)

$$\frac{\mu_i(x, y)}{\mu_i(x', y')} = \frac{\min(x'_{12}, y'_{12}) + R_0}{\min(x_{12}, y_{12}) + R_0} \qquad i = 7, \ldots, 12.$$

For $x' \in \mathrm{d}(x)$ we have

$$\begin{aligned} x'_{12} + R_0 &= |x_{12} + (x'_1 - x_1) + (x_2 - x'_2)| + R_0 \\ &\geq (1 - 2b_1)(x_{12} + R_0). \end{aligned} \tag{4.11}$$

Analogously

$$x'_{12} + R_0 \leq (1 + 2b_1) x_{12} + R_0. \tag{4.12}$$

For $x' \in \mathrm{d}(x)$, $y' \in \mathrm{d}(y)$ this yields the estimate (3.4) with $\varkappa_1 = (1 + c_1)/(1 - c_1)$, $\varkappa_2 = (1 - 3c_1)/(1 - c_1)$.

Denote

$$\gamma_0 = k/R_0 \tag{4.13}$$

this value will be supposed to be sufficiently small which is just the case for a WR motion.

In view of (4.8) and (4.9) the conditions for S can be written as

$$|\dot{x}_p(s)| \leq 2c_1 \tag{4.14}$$

$$|\dot{v}_p(s)| \leq kq_0(x_{12}(0) + R_0)^{-2} \tag{4.15}$$

$$\sum_{p=1,2} \{|\dot{\boldsymbol{v}}_p(s)-\dot{\boldsymbol{v}}_p(s')|+(x_{12}(0)+R_0)^{-1}|\dot{\boldsymbol{x}}_p(s)-\dot{\boldsymbol{x}}_p(s')|\}$$
$$\leq Q_0(x_{12}(0)+R_0)^{-2}|s-s'| \qquad \text{for } s,\ s'\in J_{X(0)}. \tag{4.16}$$

On performing calculations using the explicit form of $\boldsymbol{h}_p^*(X_t)$ and lemma 3 one obtains

$$|\boldsymbol{h}_p^*(X)|\leq C'k(x_{12}(0)+R_0)^{-2} \tag{4.17}$$

$$\sum_{p=1,2} |\boldsymbol{h}_p^*(X)-\boldsymbol{h}_p^*(X')|$$
$$\leq (k/\bar{\Delta}(x,x'))C\sum_{p=1,2}\sup\{|\delta\dot{\boldsymbol{v}}_p(s)|+|\delta\dot{\boldsymbol{x}}_p(s)|/\bar{\Delta}(X(0),X'(0))$$
$$+|\delta\boldsymbol{x}_p(s)|/[\bar{\Delta}(X(0),X'(0))]^2,\ |s|\leq\Delta(X(0),X'(0))\} \tag{4.18}$$

where $C=C_0(1+Q_0+\gamma_0q_0)$, $C'=C_0'(1+\gamma_0q_0)$, C_0, C_0' being the numerical constants $=0(1)$ for $\gamma_0\to0$ defined only by the functions $\boldsymbol{K}_1$, $\boldsymbol{K}_2$.

It is also easy to see that

$$|\boldsymbol{p}(\boldsymbol{v})|\leq c_1 \tag{4.19}$$

$$|\boldsymbol{p}(\boldsymbol{v})-\boldsymbol{p}(\boldsymbol{v}')|\leq 3|\boldsymbol{v}-\boldsymbol{v}'|. \tag{4.20}$$

By virtue of (4.17) and (4.19) the estimate (3.17) is fulfilled on the condition that

$$\begin{aligned} &g_i(x)=kC'(x_{12}+R_0)^{-2} \qquad i=1,\ldots,6\\ &g_i(x)=c_1 \qquad i=7,\ldots,12 \end{aligned} \tag{4.21}$$

Inequality (3.2) relating $g_i(x)$ and $g_i^1(x)$ is avalid if

$$C_0'(1+\gamma_0q_0)<q_0.$$

In previous estimates q_0 has not been specified. For sufficiently small $\gamma_0(\gamma_0C_0'<1)$ one can choose

$$q_0=C_0'/(1-C_0'\gamma_0) \tag{4.22}$$

in order that (3.2) be valid.

For (3.3) to be valid we put

$$\begin{aligned} &g_i^0(x)=[(1-c_1)/(1-3c_1)]^2g_i(x) \qquad i=1,\ldots,6\\ &g_i^0(x)=g_i(x) \qquad i=7,\ldots,12 \end{aligned} \tag{4.23}$$

then (3.3) follows from the estimates analogous to (4.11) and (4.12). Inequalities (3.4) agree with the choice of $\varkappa_1$, $\varkappa_2$. Inequality (3.18) is the consequence of (4.18), (4.20) and (4.9), provided that

$$L^0(x,x')=3[\bar{\Delta}(x,x')]^{-1} \tag{4.24}$$

$$L^1(x,x')=\gamma_0C[\bar{\Delta}(x,x')]^{-1} \tag{4.25}$$

$$L^2(x,x')=\gamma_0C. \tag{4.26}$$

Define $L(x,x')$ as

$$L(x,x')=Q_1[\bar{\Delta}(x,x')]^{-1} \tag{4.27}$$

then (3.9) is valid with $\sigma=(1-c_1)/(1-3c_1)$.

Note that the constants C_0, C_0', C_1 were defined in view of the structure of the SFDE (4.6) and they do not depend on the choice of γ_0. The constant Q_0, Q_1 and γ_0 have not yet been fixed. Substituting (4.24)-(4.27) into conditions (3.10), (3.21) and (3.22) one sees that they are valid for $\gamma_0 \to 0$ on the condition that

$$Q_1 > 3 + \mathrm{O}(\gamma_0) \qquad Q_0 > 3\sigma_0 Q_1 c_1 + \mathrm{O}(\gamma_0).$$

Therefore all the conditions of theorem 2 are fulfilled for sufficiently small γ_0. As the result we have the next lemma.

Lemma 4. There exists $\varepsilon > 0$ and the Lipschitz continuous functions $\boldsymbol{G}_p$ such that any solution of (2.4) satisfying (4.7) is the solution of (2.2).

Proof. The statement of the lemma follows from theorem 2 in view of the considerations of this section: there exist an SODE (3.24), its solutions satisfying the auxiliary equations (4.6). By construction of the mapping $\boldsymbol{A}$ in this case this SODE has the form

$$\dot{\boldsymbol{v}}_p = \boldsymbol{G}_p(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{x}_1, \boldsymbol{x}_2) \qquad \dot{\boldsymbol{x}}_p = \boldsymbol{p}(\boldsymbol{v}_p) \qquad p = 1, 2. \tag{4.28}$$

By virtue of theorem 2 the RHS of the SODE $\in \boldsymbol{M}$; this yields the Lipschitz continuity of $\boldsymbol{G}_p$ in $\mathbb{R}^{12}$ due to (3.8) and the expression (4.9) for $\mu_i(y, y')$ and (4.27) for $L(y, y')$. The assertion of the lemma then follows from the definition of $\boldsymbol{p}(\boldsymbol{v})$. □

4.3. *The transition from truncated equations to (2.2)*

To return to (2.2) we show that the solutions of the truncated equations corresponding to the initial data from $D(\varepsilon)$ are indeed WR trajectories satisfying (2.2).

Further calculations use the following identities:

$$\begin{aligned} \boldsymbol{x}_p(t-r) &= \boldsymbol{x}_p(t) - r\dot{\boldsymbol{x}}_p(t) - r^2 \int_0^1 \mathrm{d}s(1-s)\ddot{\boldsymbol{x}}_p(t-rs) \\ \dot{\boldsymbol{x}}_p(t-r) &= \dot{\boldsymbol{x}}_p(t) - r\int_0^1 \mathrm{d}s\, \ddot{\boldsymbol{x}}_p(t-rs) \qquad p = 1, 2 \end{aligned} \tag{4.29}$$

and the identity for the function $f(P)$ from (2.2′):

$$f(P) = 1 + (P-1)\int_0^1 \mathrm{d}s f'(1 + (P-1)s) \tag{4.30}$$

where $f' = \mathrm{d}f/\mathrm{d}P$.

Using these we separate the Newtonian limit from the terms corresponding to $\boldsymbol{K}_1$ in (4.2). The remaining terms are: (a) the velocity-dependent terms defined by the instantaneous values $\dot{\boldsymbol{x}}_p$ and $\boldsymbol{x}_p$; (b) the functional terms containing all the functional dependence upon the prehistory due to the second derivatives in (4.29). The terms (a) can be estimated as $\leqslant k\mathrm{O}(\varepsilon)[x_{12}(t)]^{-2}$ for $|\dot{\boldsymbol{x}}_p(t)| \leqslant \sqrt{\varepsilon}$ and the terms (b) are estimated in analogy to (4.4). After lengthy calcultions both groups of terms equally with those terms corresponding to $\boldsymbol{K}_2$ can be gathered into $\boldsymbol{H}_p^{(1)}$, where

$$\boldsymbol{H}_p(X_t) = \frac{g\boldsymbol{x}_{pq}(t)}{m_p[x_{pq}(t)]^3} + \boldsymbol{H}_p^{(1)}(X_t) \tag{4.31}$$

the second item satisfying the inequality

$$\boldsymbol{H}_p^{(1)}(X_t) \leqslant C_2 \varepsilon k[x_{12}(t)]^{-2} \tag{4.32}$$

provided that $\{\dot{x}_1, \dot{x}_2, x_1, x_2\} \in S$ and

$$\{\dot{x}_1(t), \dot{x}_2(t), x_1(t), x_2(t)\} \in D(\varepsilon)$$

here $C_2 = 0(1)$ as $\varepsilon \to 0$.

Lemma 5. Let the SODE (2.4) be defined by lemma 4. There exist ε, $\varepsilon'(\varepsilon > \varepsilon' > 0)$ such that any solution which is specified by the initial data (2.3) from $D(\varepsilon')$, does not leave $D(\varepsilon)$.

Proof. Define

$$E(t) < \tfrac{1}{2} \sum_{p=1,2} m_p[\dot{x}_p(t)]^2 + g[x_{12}(t)]^{-2}. \tag{4.33}$$

Any solution of (4.28) $(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{x}_1, \boldsymbol{x}_2) \in S$ by virtue of lemma 1. Let the initial data be chosen in such a way that

$$E(t_0) = (\varepsilon'/2)\min(m_1, m_2) \qquad 0 < \varepsilon' < \varepsilon$$

ε being sufficiently small. This inequality is valid on some open interval $(t_0', t_0'') \ni t_0$ owing to continuity arguments. The solution satisfies (4.6) and, for a sufficiently small ε', $\dot{x}_p = v_p$ and (x_1, x_2) satisfies (2.2). Using (2.2) and (4.31) one obtains on this interval

$$\ddot{x}_{12}(t) \geq (x_{12}(t)\ddot{x}_{12}(t))/x_{12}(t)$$

$$\geq \frac{g}{m_r[x_{12}(t)]^2} - \sum_{p=1,2} |H_p^{(1)}(X_t)| \tag{4.34}$$

$$m_r = m_1 m_2/(m_1 + m_2)$$

and by virtue of (4.32)

$$\ddot{x}_{12}(t) \geq (1 - C_3\varepsilon) g[x_{12}(t)]^{-2}/m_r \tag{4.35}$$

where $C_3 = C_2 m_r/\min(m_1, m_2)$.

From (4.33) using (2.2)

$$\frac{dE}{dt} = \sum_{p=1,2} m_p(\dot{x}_p H_p^{(1)}) \leq \sum_{p=1,2} m_p|\dot{x}_p||H_p^{(1)}| \tag{4.36}$$

(on account of (4.32) and (4.35)).

Evidently

$$\sum_{p=1,2} m_p|\dot{x}_p| \leq \left\{M \sum_{p=1,2} m_p \dot{x}_p^2\right\}^{1/2} < (2ME)^{1/2} \qquad M = m_1 + m_2$$

then from (4.36) on account of (4.35) one obtains

$$d(E^{1/2})/dt \leq C_2\sqrt{\frac{M}{2}}\,\varepsilon k/x_{12}^2 \leq \sqrt{\frac{M}{2}}\,\frac{C_2\varepsilon m_r \ddot{x}_{12}}{(1 - C_2\varepsilon)\min(m_1, m_2)}$$

whence

$$E^{1/2}(t) \leq E^{1/2}(t_0) + \sqrt{\frac{M}{2}}\,\frac{C_2\varepsilon m_r(|\dot{x}_{12}(t)| + |\dot{x}_{12}(t_0)|)}{(1 - C_2\varepsilon)\min(m_1, m_2)}$$

$$\leq E^{1/2}(t_0) + C_3\varepsilon[E^{1/2}(t) + E^{1/2}(t_0)]$$

$C_3 = 0(1)$ for $\varepsilon \to 0$.

Therefore on (t_0', t_0'') we have

$$E^{1/2}(t) \leqslant E^{1/2}(t_0)(1+C_3\varepsilon)/(1-C_3\varepsilon).$$

For a sufficiently small ε

$$(C_3\varepsilon < 1, (1+C_3\varepsilon)\sqrt{\varepsilon'}/(1-C_3\varepsilon) < \sqrt{\varepsilon})$$

we see that the solution does not leave $D(\varepsilon)$ on $[t_0', t_0'']$ and in the neighbourhood of t_0', t_0''. Then the principle of the continuous induction (see e.g. [13]) extends the assertion of the lemma on $\forall t$. □

On account of lemma 5 we see that the solutions of the SODE (2.4) satisfying (2.3) with the data from $D(\varepsilon')$ satisfy the SFDE (2.2). This proves the first part of theorem 1.

4.4. *Uniqueness of solutions of (2.2) with conditions (2.3)*

In order to use the estimates on the RHS of (2.2) we need the following lemma.

Lemma 6. For a sufficiently small $\varepsilon > 0$ any solution

$$\{\boldsymbol{x}_1, \boldsymbol{x}_2\} \in W(\varepsilon)$$

of SFDE (2.2) satisfies the relation

$$\{\dot{\boldsymbol{x}}_1, \dot{\boldsymbol{x}}_2, \boldsymbol{x}_1, \boldsymbol{x}_2\} \in \boldsymbol{S}.$$

Proof. The relation to $\boldsymbol{S}$ is defined by (4.14)-(4.16). The first inequality follows from the definition of $W(\varepsilon)$. To obtain the others, (2.2) (cf (4.1)) should be written as the linear system with respect to $z = (z_1, \ldots, z_6) = (\ddot{\boldsymbol{x}}_1, \ddot{\boldsymbol{x}}_2)$

$$z_i(t) = A_i(t) + \sum_{j=1}^{6} \{B_{ij}(t) z_j(t) + C_{ij}(t) z_j(t - r_{12}(t)) + D_{ij}(t) z_j(t - r_{21}(t))\} \qquad i = 1, \ldots, 6 \tag{4.37}$$

where A_i, B_{ij}, C_{ij}, D_{ij} are continuously differentiable functions satisfying the inequalities

$$|A_i(t)| \leqslant kC_4[x_{12}(t)]^{-2} \tag{4.38}$$

$$\sum_{i,j} \{|B_{ij}(t)| + |C_{ij}(t)| + |D_{ij}(t)|\} \leqslant kC_5/x_{12}(t). \tag{4.39}$$

From the definition of $W(\varepsilon)$ it follows that $|z_i(t)|$ is bounded. Estimating $x_{12}(t)|z_i(t)|$ in (4.37) on account of (4.38) and (4.39) we obtain

$$\sup\{x_{12}(t)|z_i(t)|, t \in \mathbb{R}\} < \infty.$$

Using lemma 3 we have

$$x_{12}(t - r_{pq}) = x_{12}(t) - 2\sqrt{\varepsilon}\, r_{pq}(t) \geqslant \frac{1 - 3\sqrt{\varepsilon}}{1 - \sqrt{\varepsilon}} x_{12}(t).$$

then

$$x_{12}(t)|z_i(t - r_{pq}(t))| \leqslant \frac{1 - \sqrt{\varepsilon}}{1 - 3\sqrt{\varepsilon}} \sup\{x_{12}(t)|z_i(t)|, t \in \mathbb{R}\}.$$

Using this in (4.37) multiplied by $[x_{12}(t)]^2$ we obtain

$$\exists M_0 = \sup\{[x_{12}(t)]^2|z_i(t)|,\ t\in\mathbb{R}\} \leqslant kC_4 + 3kC_5\frac{1-\sqrt{\varepsilon}}{1-3\sqrt{\varepsilon}}\sup\{x_{12}(t)|z_i(t)|,\ t\in\mathbb{R}\}[x_{12}(t)] < \infty. \tag{4.40}$$

Again from (4.37) using $k/x_{12}(t)\leqslant\varepsilon$ one obtains

$$M_0 \leqslant kC_4 + 3\varepsilon\left(\frac{1-\sqrt{\varepsilon}}{1-3\sqrt{\varepsilon}}\right)^2 C_5 M_0.$$

This yields the estimate for M_0 corresponding to (4.15) provided that

$$3\varepsilon\left(\frac{1-\sqrt{\varepsilon}}{1-3\sqrt{\varepsilon}}\right)^2 C_5 < 1.$$

The estimate (4.16) involving $w_i = z_i(t+s) - z_i(t+s')$ is obtained by means of analogous (but somewhat more cumbersome) considerations using the equation for w_i following from (4.37). □

Now we shall use the fact that all the trajectories defined by (2.2), are asymptotically free (AF), i.e. they pass through the region of nearly uniform motion.

The solution of (2.2) will be said to be AF if $\exists R>0$, $u>0$ and T_0 such that for $t<T_0$

$$|\boldsymbol{x}_{12}(t)| \geqslant R + u|t-T_0|. \tag{4.41}$$

Lemma 7. For a sufficiently small $\varepsilon>0$ all the solutions $(\boldsymbol{x}_1, \boldsymbol{x}_2)\in W(\varepsilon)$ of the SFDE (2.2) are AF with common constants R, u for given data (2.3).

Proof. For the solutions from $W(\varepsilon)$ the consideration is the same as for (4.6) in lemma 5. Then (4.41) follows from (4.35), for example making use of the analogous ID considerations of [5-8]. □

The final step deals with the proof that arbitrary AF solutions of (2.2) are unique under conditions (2.3) for sufficiently large initial separations of the particles, and, therefore, that any AF solution $\{\boldsymbol{x}_1, \boldsymbol{x}_2\}\in W(\varepsilon)$ satisfies (2.4).

It is clear that in (4.41) R may be considered to be sufficiently large with an appropriate choice to T_0, u remaining fixed. Denote

$$\boldsymbol{v}_{p,0} = \dot{\boldsymbol{x}}_p(T_0) \qquad \boldsymbol{x}_{p,0} = \boldsymbol{x}_p(T_0) \qquad |\boldsymbol{x}_{1,0} - \boldsymbol{x}_{2,0}| > R \tag{4.42}$$

and suppose that (4.42) is valid for two AF solutions x, x' of (2.2). Denoting $\delta\boldsymbol{x}_p = \boldsymbol{x}_p - \boldsymbol{x}'_p$ we have

$$\delta\boldsymbol{x}_p(T_0) = 0 \qquad \delta\boldsymbol{v}_p(T_0) = 0 \tag{4.43}$$

where $\delta\boldsymbol{v}_p = \delta\dot{\boldsymbol{x}}_p$; then for $t\leqslant T_0$ it is easy to see that

$$|\delta\boldsymbol{x}_p(t)| = (\tfrac{1}{2})|t-T_0|^2 A_p \tag{4.44}$$

$$|\delta\boldsymbol{v}_p(t)| = |t-T_0| A_p \tag{4.45}$$

where $A_p = \sup\{|\delta\dot{\boldsymbol{v}}_p(s)|,\ s\leqslant T_0\}$

$$\frac{T_0 - t + \Delta(x(t), x'(t))}{\bar{\Delta}(x\cdot x')} \leqslant \frac{T_0 - t}{u(T_0-t)+R} + 1 < \frac{1}{u} + 1. \tag{4.46}$$

For these solutions from $W(\varepsilon)$ the estimates analogous to (4.17) and (4.18) are valid, whence by virtue of (2.2) and making use of (4.44)–(4.46) one obtains

$$|\delta\dot{v}_p(t)| \leqslant \frac{kC}{\bar{\Delta}(x(t), x'(t))} \sum_{p=1,2} \sup\left\{ |\delta\dot{v}_p(s)| + \frac{|\delta v_p(t)|}{\bar{\Delta}(x(t), x'(t))} + \frac{|\delta x_p(t)|}{[\bar{\Delta}(x(t), x'(t))]^2}, \right.$$

$$\left. s \in [t - \Delta(x(t), x'(t)), t] \right\}$$

$$\leqslant \frac{kC}{R}(2 + 3/u + 1/u^2) \sum_{p=1,2} A_p.$$

For a sufficiently large R this means $\delta x_p \equiv 0$. □

Thus the solution considered in the last lemma is unique. Therefore it is just the solution of the SODE (4.28) which is specified uniquely by the initial data and which satisfies all necessary restrictions on the class of functions. This ends the proof of theorem 1.

5. Discussion

The results obtained in this paper show that the presence of the functional hereditary terms in EOMs does not conflict with the existence of an 'instantaneous' form (2.4) describing all the WR solutions, if one assumes that the material system in question was isolated in the past (i.e. that the EOMs must be valid up to $t \to -\infty$). The important point of our investigation is the consideration of the function class in which the solutions are sought, which singles out the WR phase flow. The existence of such phase flows is probably the general property of few-body WR hereditary systems, though they may not exhaust all the physical trajectories. On the other hand, the assertion that it is impossible to express the WR solutions of the physical system in terms of its instantaneous state (cf [3]) would mean the presence of the intersections of the corresponding phase trajectories. In order to prove such an assertion the nonuniqueness of trajectories satisfying the 'instantaneous' initial data should be demonstrated.

The approximations of (2.4) are generated by iterations of (3.20). One can show that the first iterations correspond to the approximate EOMs obtained formally using some kinds of expansions in powers of the interaction constant g.

Note that from the group properties of initial EOMs (2.2) it follows that the exact RHS of SODE (2.4) satisfies the Currie–Hill conditions of relativistic invariance [14, 15].

The present consideration does not include the case $g < 0$. However, if the solutions of the EOMs are AF for $t \to -\infty$ according to (4.41), the estimates of section 4 will be applicable. It seems that this is just the case if one brings into play dissipative effects (due to radiation processes, etc.): in the presence of energy gain for the time-reversed EOMs one expects that the two-body system will expand for $t \to -\infty$, the main portion of WR trajectories being AF.

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